A Dynamical Approach to Fractional Brownian Motion

Riccardo Mannella

Dipartimento di Fisica dell'Università di Pisa

Piazza Torricelli 2, 56100 Pisa, Italy

Paolo Grigolini[1]

Istituto di Biofisica del Consiglio Nazionale delle Ricerche

Via San Lorenzo 28, 56127 Pisa, Italy

Bruce J. West

Department of Physics, University of North Texas

P.O. Box 5368, Denton, Texas 76203

Abstract

Herein we develop a dynamical foundation for fractional Brownian Motion. A clear relation is established between the asymptotic behaviour of the correlation function and diffusion in a dynamical system. Then, assuming that scaling is applicable, we establish a connection between diffusion (either standard or anomalous) and the dynamical indicator known as the Hurst coefficient. We argue on the basis of numerical simulations that although we have been able to prove scaling only for "Gaussian" processes, our conclusions may well apply to a wider class of systems. On the other hand systems exist for which scaling might not hold, so we speculate on the possible consequence on the various relations derived in the paper on such systems.

[1]Also at: Dipartimento di Fisica dell'Università di Pisa, Piazza Torricelli 2, 56100 Pisa, Italy, and Department of Physics of the University of North Texas, P.O. Box 5368, Denton, Texas 76203

1 Introduction

A stochastic process is in general characterized by two quantities, the probability density describing the random nature of the fluctuations and the correlation function describing how a fluctuation at a given time influences subsequent fluctuations. The statistics and the spectrum of the random variations are therefore independent quantities and both are necessary to completely describe a stochastic process. For example, a Gaussian distribution may specify the statistics of a process, but the time dependence of the variance, i.e. the width of the distribution, depends on the correlation function or spectrum of the process. Given that in many physical systems the Gaussian distribution is a straightforward consequence of the Central Limit Theorem, investigators are often satisfied with measurements of the correlation function to describe natural stochastic processes particularly in the observation of large scale phenomena such as in geophysics. One such scientist who was most successful in this regard was Hurst, who was particularly interested in problems of Hydrology and the Nile river [1, 2].

Hurst [1, 2, 3] developed a method called rescaled range analysis, in which the span of a random process is devided by its variance, resulting in a new variable that depends on the time length of the data record in a particularly interesting way. Let us define the time average of the random process $\xi(t)$ over the interval of time τ: introducing t, the discrete integer-valued time at which the observations are recorded, and τ, the total integer valued time-interval considered, we have

$$\langle \xi \rangle_\tau = \frac{1}{\tau} \sum_{t=1}^{\tau} \xi(t) \tag{1}$$

Let us also define $X(t)$, the "accumulated departure" of $\xi(t)$ from the mean $\langle \xi \rangle_\tau$, i.e.

$$X(t,\tau) \equiv \sum_{u=1}^{t} \xi(u) - \langle \xi \rangle_\tau \tag{2}$$

so that the span of the process is defined by

$$S(\tau) = \max_{1 \leq t \leq \tau} X(t,\tau) - \min_{1 \leq t \leq \tau} X(t,\tau) \tag{3}$$

Finally, let us also consider the standard expression for the variance

$$V(\tau) = \left[\frac{1}{\tau} \sum_{t=1}^{\tau} (\xi(t) - \langle \xi \rangle_\tau)^2 \right]^{\frac{1}{2}} \tag{4}$$

The rescaled Hurst analysis consists in studying the properties of the ratio

$$R(\tau) = S(\tau)/V(\tau) \tag{5}$$

He found that for the time records of over 850 phenomena R is well described by the following empirical relation

$$R(\tau) = (\tau/2)^{H_H} \tag{6}$$

Here we refer to H_H as the Hurst exponent (Hurst used the symbol K for this exponent).

Mandelbrot and co-workers [4, 5, 6] modeled this heuristic result using the theory of fractional Brownian motion. They showed that fractional Brownian motion would provide an explicit statistical realization of (6), and that the theory would imply a reasonable interpretation of the parameter H_H. We emphasize that this important result supports the interpretation of natural phenomena in terms of fractal functions. However, this interpretation does not take into account the fact that fractal functions, as important as they may be, are still idealizations of natural phenomena. These idealizations are not applicable to all time scales. In this context we mention the recent efforts to establish the physical foundation of classical statistical mechanics using the concept of chaos [7, 8, 9].

These efforts rely on there being a wide time scale separation between the microscopic and the macroscopic dynamical regimes. The microscopic quantities of motion are valid on a very short time scale, where the conventional statistical arguments are inapplicable. Then, upon increasing of the time scale considered, as a result of the action of chaos, the system of interest exhibits those statistical properties which are associated with the conventional ideas of a canonical distribution and Gaussian statistics.

The main purpose of the present paper is to provide a dynamical basis for the Hurst rescaled range analysis. We show that the theory of Mandelbrot and co-workers [4, 5, 6] focuses on the asymptotic limit of the dynamical processes considered herein.

There are a number of benefits arising from this change in perspective. First of all, adding a "dynamical dimension" to the Hurst analysis, can be profitably used to quantitatively illustrate the evolution of a deterministic system away from a regular toward a totally chaotic process, the latter being, for many purposes, virtually indistinguishable from a stochastic one. Thus, the Hurst exponent may provide an independent means of distinguishing stochastic from chaotic motion [10]. Following this idea, we are tempted to speculate that the distinction between these two kinds of physical processes may be merely a question of time scale.

Secondly we shall see that such a dynamical analysis, i.e., putting the fractal geometry of Mandelbrot into a dynamical setting, has the beneficial effect of rigorously establishing a connection between the Hurst coefficient and the behaviour of the autocorrelation functions at long times. We link the coefficient to the existence of positive or negative tails for the autocorrelation function of the random variable ξ,

$$C(t) = \langle \xi(0)\xi(t) \rangle \tag{7}$$

In molecular dynamics, dynamical processes characterized by a long-time regime with an inverse power-law correlation function

$$C(t) = \pm k/t^{\alpha} \qquad \text{for} \quad t \to \infty \tag{8}$$

are denoted as *slow-decay* processes since the pioneering work of Adler and Wainwright [11] where (8) was first identified, correlation functions have been the subject of an intense debate [12, 13]. From (8) we note that the power law decay can have either a positive or a negative tail. We shall see that the processes that Mandelbrot denotes as *persistent*, $H_H > 1/2$ in (6), are connected by our theoretical analysis to a positive tail, whereas those termed by him to be *antipersistent*, $H_H < 1/2$ in (6), are connected by our theoretical analysis to the existence of negative tails.

We support the analytic arguments presented in Section 2 and 3 with computer calculations done for a substantial number of different dynamical systems. The numerical results presented in Section 4 support the suggested dynamical approach to the "geometrical" theory of Mandelbrot, but they also suggest that the asymptotic time regime itself might be explored with different mathematical arguments, valid also for non-Gaussian statistics.

2 Dynamical theory

Let us now focus our attention on the following equation of motion

$$\dot{x} = \xi \tag{9}$$

The Hurst coefficient (6) was defined in terms of a discrete time process so any dynamical representation such as (9) ought to be discrete as well. However, to connect the process

with the field of molecular dynamics, we adopt a continuous time representation. The formal time integration of (9) yields

$$x(t) = \int_0^t \xi(t')dt' + x(0) \tag{10}$$

Thus the variable $x(t)$ undergoes a kind of motion with erratic fluctuations induced by $\xi(t)$. Later on we shall define more precisely the nature of such "disordered" motion. For the time being we limit ourselves to the conventional language of statistical mechanics. Thus, rather than focusing on single trajectories we shall study the mean values of quantities like $x^n(t)$. We make the simplifying assumption that the erratic variable $\xi(t)$ fluctuates around zero. Thus $\langle x(t)\rangle = \langle x(0)\rangle$ where the brackets denotes an average over an ensemble of realisations of $\xi(t)$, as well as the distribution of initial state of $x(t)$. We are therefore obliged to study the mean value of $x^2(t)$. By averaging $x^2(t)$ over the fluctuations and initial conditions, we obtain

$$\langle x^2(t)\rangle = \int_0^t dt' \int_0^t dt'' \langle \xi(t')\xi(t'')\rangle + 2\int_0^t dt' \langle \xi(t')x(0)\rangle + \langle x^2(0)\rangle \tag{11}$$

We make the assumption that the second term on the r.h.s. of (11) vanishes, hypothesizing no correlation between the initial value of x and ξ. This assumption certainly holds true when there exists a large time scale separation between the dynamics of the *fast variable* ξ and that of the *slow variable* x. However, this hypothesis must be used with caution in applications, which often refer to situations of slow decay (8), implying an exceptionally extended memory. We then adjust the limits of the time integrals and obtain

$$\langle x^2(t)\rangle = 2\int_0^t dt' \int_0^{t'} dt'' \langle \xi(t')\xi(t'')\rangle + \langle x^2(0)\rangle \tag{12}$$

Under the assumption that the process $\xi(t)$ is stationary, i.e., its moments are inde-

pendent of the time origin, so that

$$\langle \xi(t')\xi(t'')\rangle = \langle \xi(t'-t'')\xi(0)\rangle \tag{13}$$

we derive from (12) the following integro-differential equation of motion

$$\frac{d}{dt}\langle x^2(t)\rangle = 2\int_0^t \langle \xi(\tau)\xi(0)\rangle d\tau \tag{14}$$

Clearly, the $\langle x^2(t)\rangle$ appearing in the l.h.s. of (14) must be connected to the long-time diffusional regime described by

$$\langle x^2(t)\rangle = Kt^{2H_D}. \tag{15}$$

It is evident that the physical bounds on the possible values of H_D are given by

$$0 < H_D < 1; \tag{16}$$

$H_D = 0$ defines the case of localization, which is the lower limit of any diffusion process, and $H_D = 1$ obviously refers to the case of many uncorrelated deterministic trajectories, with $x(t) - x(0)$ linearly proportional to time for each of them. The bound $H_D < 1$ is a consequence of the fact that a diffusional process cannot spread faster than a collection of deterministic trajectories! Finally, the condition $H_D = 1/2$ is obtained for simple Brownian motion, where the variance increases linearly with time.

On the other hand, using the definition of the correlation function given in (7), (14) becomes

$$2D(t) \equiv \frac{d}{dt}\langle x^2(t)\rangle = 2\int_0^\infty C(t')dt'. \tag{17}$$

We can now show that, using (17), the deviation of H_D from the conventional diffusion prediction $H_D = 0.5$ can be explained if the correlation function $C(t)$ exhibits a slow decay. The joint use of (8), (15) and (17) leads to the following long-time prediction

$$\frac{d^2}{dt^2}\langle x^2(t)\rangle = 2H_D(2H_D-1)Kt^{2H_D-2} \sim 2C(t) = \pm\frac{2k}{t^\alpha} \tag{18}$$

having assumed that the long-time limit of the correlation function $C(t)$ is dominated by the inverse power law of (8). The positive (negative) sign refers to the case of the solid (dashed) line in Fig. 1.

From (18) we determine that H_D and α satisfy the relation

$$H_D = 1 - \alpha/2 \tag{19}$$

obtained by matching the time dependences. It is also clear from the coefficients in (18) that $H_D > 1/2$ implies a positive long-time correlation, whereas $H_D < 1/2$ implies a negative long-time correlation. Let us now summarize the result of this simple theoretical analysis, with an eye to Fig. 1:

$$\begin{array}{rl} \text{Case exemplified by the solid line:} & 1 > H_D > 1/2; \quad 1 > \alpha > 0 \\ \text{Case exemplified by the dashed line:} & 0 < H_D < 1/2; \quad 2 > \alpha > 1 \end{array}$$

Thus, we see that the solid line correlation function of Fig. 1 leads to a superdiffusive behavior ranging from the standard diffusion ($H_D = 1/2$) to the ballistic behavior ($H_D = 1$). The dashed line correlation function of the same figure leads to a subdiffusive behavior ranging from the standard diffusion to no motion at all.

It must be stressed that for superdiffusive correlation functions (17), $D(\infty) = \infty$, whereas in the case of subdiffusive correlation functions, $D(\infty)$ is finite or even vanishing. In this latter case we obtain what we define as *classical Anderson localization.* At early times in the diffusional process the mean square value of $x(t)$ increases. Then when the negative part of the correlation function $C(t)$ becomes important, the rate of diffusion decreases. When the negative tail completely compensates for the positive part of the relaxation process, the rate of diffusion virtually vanishes. At this late time stage further

diffusion is rigorously prevented and the diffusing particle becomes localized. Processes of this kind have recently been discovered [14] and the theory presented here affords a remarkably straightforward explanation of them. It is interesting that such processes should admit such a simple interpretation.

We point out that in the simple theory presented in this section, the only significant assumptions made are the stationary property of (13) and the absence of correlation between $x(0)$ and ξ. No assumption was made on the nature of the statistics of the stochastic process x except that it has a finite correlation function. In the next section we shall rederive (14) under the more restrictive assumption that the process ξ and therefore x is Gaussian. We also note that the standard case $H_D = 0.5$ is compatible with almost any kind of relaxation process. The only condition to fulfill is that the correlation function $C(t)$ is square integrable over the time interval . Thus, if we exclude the case $H_D = 0.5$, we must invoke an inverse power law decay to explain the behavior given by (15). This is so because (15) implies the existence of "stationary" behavior and which, in turn, implies the existence of an inverse power-law decay. A power-law decay is the only way of "killing" the possibility itself of defining a time scale, and this, in turn, is a condition essential to explain the "stationary" nature of the diffusion regime of (7) with $H_D \neq 0.5$. What do we mean by "stationary"? Upon increase of the time scale considered, the diffusion process is increasingly dependent on the tail of the correlation function, until it becomes totally dominated by the inverse power-law tail. In this regime, since the power-law decay of the correlation function implies no time scale is dominant, the diffusion process becomes stationary. The concept of a stationary diffusion process can be easily expressed by referring to the Hurst coefficient rather than to H_D. We shall see that H_H is time dependent, and that it usually reaches a stationary value for $t \rightarrow \infty$.

This definition of "stationary" diffusion behavior implies that the diffusion coefficient H_D can be identified with the asymptotic value of the Hurst coefficient H_H. We shall see that this is frequently the case even if we shall only be able to *rigorously* prove it for Gaussian statistics.

3 A Fokker-Planck treatment

The next step is to determine the connection of the Hurst coefficient H_H with the diffusion coefficient H_D. To do this we need to derive suitable expressions for the quantities appearing in (5), for any given dynamical system. In part we follow the strategy of Mandelbrot: the idea is that for a white Gaussian process it is simple to carry out the theoretical analysis, hence it is only necessary to find, for a given dynamical system, the corresponding "Gaussian" approximation in the appropriate "reduced" time scale.

The details of the approach are given in reference [15]. For the given dynamical system we want to study, we replace the dynamical equations with the following equation of motion for the probability density

$$\frac{\partial}{\partial t}\rho(x,\xi,\Gamma;t) = L\rho(x,\xi,\Gamma;t). \tag{20}$$

Here Γ stands for the entire set of variables necessary to describe the time evolution of ξ. The "Liouville-like" operator L is divided into two parts as follows,

$$L = L_I + L_B \tag{21}$$

where the "interaction" part determined by (9) yields and the phase space operator

$$L_I = -\xi\frac{\partial}{\partial x}, \tag{22}$$

and L_B defines the time evolution of the distribution of the variables ξ and Γ, determined by the dynamics of the corresponding set of variables. It is not necessary to define the explicit form of the latter operator since it depends in detail on the specific problem studied.

We now use the Zwanzig projection approach [16], which consists in integrating the total distribution ρ over the degrees of freedom that are not of interest to us

$$\sigma(x,t) = \int d\xi d\Gamma \rho(x,\xi,\Gamma) \tag{23}$$

We apply this projection approach following the perturbation prescriptions of [14], assuming that L_I in (22) is a "weak" perturbation. Using this basic assumption, after some algebra described in detail in [15], we arrive at the result

$$\frac{\partial}{\partial t}\sigma(x,t) = \Xi(t)\frac{\partial^2}{\partial x^2}\sigma(x,t), \tag{24}$$

where

$$\Xi(t) = \int_0^t C(\tau)d\tau \tag{25}$$

If (24) was used to determine the time evolution of $\langle x^2(t)\rangle$ i.e., multiply (24) by x^2 and integrate over x, it would reduce to (17). In this sense (17) and (24) are equivalent. Unfortunately, the two equations are not truely equivalent because (17) is obtained without making any assumption regarding the statistics of x, whereas (24) is really the result of a second-order perturbation treatment, equivalent to assuming that the statistical process x is Gaussian.

If now we rescale the time as follows

$$t^* = t^{2H_D}, \tag{26}$$

(24) can be written as

$$\frac{\partial}{\partial t^*}\sigma(x;t^*) = \Phi\left(t^*\right)\frac{\partial^2}{\partial x^2}\sigma(x;t^*) \tag{27}$$

where, of course,

$$\Phi\left(t^*\right) = \frac{1}{2H_D} t^{*\frac{1}{2H_D}-1}\int_0^{t^{*\frac{1}{2H_D}-1}} \frac{dt'^*}{2H_D} t'^{*\frac{1}{2H_D}-1} C\left(t'^*\right) \tag{28}$$

After some algebra it can be shown that

$$\lim_{t^*\to\infty} \Phi\left(t^*\right) = K \tag{29}$$

where K is a finite constant. It is evident that in the asymptotic time limit (27) becomes a standard (time independent) Fokker-Planck equation, and the statistical process defined in terms of the scaled variable

$$y \equiv \frac{x}{\sqrt{Kt^*}}, \tag{30}$$

becomes a Gaussian process with the distribution

$$p(y) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right). \tag{31}$$

This result means that the random function $x(t)$ is proportional to $t^{*1/2}$ or, in the original time scale, to t^{H_D}. Following Mandelbrot [4], we are then led to identify H_D with H_H.

This is the central theoretical result of this paper. In the asymptotic limit the dynamical process described by (20) becomes the fractional Brownian motion of Mandelbrot. However, in the short time regime the process can be substantially different from such a stochastic process. This is clearly illustrated by the numerical results of the following section.

We note that H_D is the coefficient appearing in (15). However, to be able to identify this coefficient with H_H it is necessary that we go through the Fokker-Planck equation

of (24), and this implies making a second-order approximation, absent in the derivation of (15). In the next section we show, using numerical methods that H_H is very close in value to H_D even in cases when (24) should not be valid. This suggests the possibility of establishing the property $H_H = H_D$ without using the second-order approximation necessary for the derivation of (24).

4 Numerical simulations

We now present numerical simulations of a number of different dynamical systems, to which the Hurst analysis is then applied. Comments on the correspondence of H_D and H_H are discussed alongside the numerical data. For clarity, the different models are presented within separate subsections.

4.1 Microscopic dynamics described by the Langevin equation

The easiest way to create a stochastic process ξ with an exponential correlation function is through the Langevin equation

$$\dot{\xi} = -\gamma\xi(t) + f(t) \tag{32}$$

where $f(t)$ is Gaussian white noise. Clearly, the relevant time scale here is given by $1/\gamma$, and we expect that for times larger than $1/\gamma$, we should find $H_H = 0.5$. This is confirmed (see Fig. 2) by the numerical simulations: the expected asymptotic regime is reached at shorter times for the case of larger γ.

A similar model is given by the multidimensional Langevin equation

$$\begin{aligned} \dot{\xi} &= w(t) \\ \dot{w} &= -\Gamma w(t) - \Omega^2\xi(t) + F(t) \end{aligned} \tag{33}$$

In the case $\Gamma \gg \Omega$, (33) becomes indistinguishable from (32) with $\gamma = \Omega^2/\Gamma$. Again, as soon as the time scale considered is larger than the typical time scale of the system, we should have $H_H = 0.5$. However, here a problem arises. It is possible to consider either the time under the correlation function or the time over which the "quasi energy" in (33) looses correlation: these two times are in general very different. By inspection of Fig. 3, it is clear that numerical simulations done for different Ω^2 and the same γ show a similar asymptotic behavior more or less in the same time region (compare the solid line and the dotdashed line in Fig. 3): whereas, when a different γ value is considered (compare the solid and dashed lines), the asymptotic regime sets in at different times. This seems to imply that the relevant time scale is the time over which the energy looses correlation.

Due to the linearity of (33), the Gaussian statistics of the stochastic force is transmitted to ξ and hence to x. Thus, the Central Limit Theorem is fulfilled and the asymptotic behavior must be characterized by $H_H = H_D = 0.5$. The parallelism between R and $\langle x^2(t)\rangle^{1/2}$ of Fig. 4 shows that the transient behavior corresponds exactly to the time it takes for the second moment to reach the stationary condition, corresponding to standard diffusion. In the case of (33) this transient time is a complex function of the parameters Γ and Ω, which we do not discuss here. But of course standard diffusion sets in over time scales which are connected to the time taken by the quasi-energy to decorrelate, hence justifying the behaviour shown in Fig. 3.

4.2 Bistable stochastic motion

The next system we consider involves the motion of a stochastic particle in a bistable potential. The dynamics can be cast in the dimensionless form

$$\dot{\xi} = \xi - \xi^3 + f(t) \tag{34}$$

where $f(t)$ is a stochastic Gaussian white force [see (32)] of intensity D. The Hurst rescaled range analysis is applied to the variable ξ. In the limit of small noise intensities D the system (34) is characterized by two different time scales: one is the time scale of the inter-well relaxation time (T_R); the other one is the time scale of the intra-well dynamics (T_K, related to the Kramers rate for the system). For $\tau \gg T_K$ the system should behave as a dichotomous random process, hence $H_H = 0.5$. Also, if applicable, in the range $T_R \ll \tau \ll T_K$ we expect $H_H = 0.5$: this is because before the Hurst analysis is able to single out the dichotomous random process (the behaviour for $\tau \gg T_K$), the condition $T_R \ll \tau$ implies that, for the relevant τ values, the dynamics is similar to that of (32). This is confirmed by Fig. 5, where we show the result for $R(\tau)$ obtained by digitally simulating (34). For the simulations we use $D = 0.1$, which yields a T_K of approximately 30, and a T_R, determined by the coefficients appearing in the force in (34), of order 1. It is possible to note that for larger τ values H_H approaches the correct value 0.5, and that for $T_R < \tau < T_K$, H_H is smaller than it is, say, in the region $\tau \sim T_K$. Unfortunately, the condition $T_R \ll \tau \ll T_K$ is only weakly satisfied by our choice of the parameter D and it is not possible to observe exactly $H_H = 0.5$ in the range of intermediate τ.

4.3 Noisy Lorenz model

Another model to which we have applied the Hurst rescaled range analysis is the Lorentz model perturbed by Gaussian white noise [17]. The perturbed Lorenz model is described by the stochastic differential equations

$$\begin{aligned} \dot{\xi} &= \sigma(\xi - y) + f(t) \\ \dot{y} &= r\xi - y - \xi z \\ \dot{z} &= -z + \xi y \end{aligned} \tag{35}$$

where $f(t)$ is a Gaussian white process with intensity D. We set $\sigma = 8/3$, $b = 10$ and $r = 126.5$, a set of parameters for which the Lorenz model is known to be periodic, as long as $D = 0$. The chosen value of the control parameter $r = 126.5$ is roughly in the middle of a small periodic island within an r region for which the Lorenz model displays chaos. The addition of a small stochastic force (i.e., $D \neq 0$) "kicks" the system out of the periodic region, and leads to the observation of chaos [17]. The transition periodic/chaotic motion as function of the noise intensity D is very smooth, and chaos, defined by a positive Lyapunov exponent, is observed for $D > 10^{-5} - 10^{-4}$.

Applying the Hurst rescaled range analysis to the variable ξ of the Lorenz system, we expect that H_H will vary from around zero (the value for periodic motion) to 0.5 (diffusive dynamics) for increasing noise. Also, the τ values for which the asymptotic behaviour is observed should become smaller as D is increased. The result of numerical simulations of (35) is shown in Fig. 6. Note that in Fig. 6 we have plotted $R(\tau)/\tau^{1/2}$ versus τ, to more clearly show the asymptotic behaviour. For all curves, $R(\tau)$ increases for small τ, up to $\tau \sim 1$, which is the period of the noiseless Lorenz system for the chosen parameters. In the case of very small noise intensities (full and dotted line), we have that for larger

τ, $R(\tau)$ goes like τ^0; behaviour typical of periodic motion. When we increase the noise intensity (small dashes line), after an initial decrease, $R(\tau)$ increases: it eventually leads to $H_H = 0.5$, for still larger τ values, but on the τ range shown the asymptotic behaviour is not yet established. For this D value, we remark, that the Lorenz model is only very weakly chaotic. Finally, for even larger noise intensities (large dashes line) the departure of $R(\tau)$ from the noiseless curve takes place at yet smaller τ values, and the expected asymptotic behaviour ($H_H = 0.5$, on the figure, represented by an horizontal line) for $R(\tau)$ is clearly identified.

4.4 A two-dimensional potential

We next study the deterministic motion of a particle in the two-dimensional potential

$$V(\xi, y) = \cos\left(\xi + y\sqrt{3}\right) + \cos\left(\xi - y\sqrt{3}\right) + \cos\left(2y/\sqrt{3}\right) \tag{36}$$

which defines an infinite lattice of triangular symmetry (see [18] for more details). The equations of motion we have integrated have the form

$$\begin{aligned} \dot{\xi} &= v \\ \dot{v} &= -\frac{\partial V(\xi, y)}{\partial \xi} \\ \dot{y} &= w \\ \dot{w} &= -\frac{\partial V(\xi, y)}{\partial y} \end{aligned} \tag{37}$$

It is known that when the energy of the system is efficiently small the particle moves from the bottom of each triangular cell, only occasionally wandering from cell to cell. As a consequence, the self-diffusion coefficient shows a peculiar behaviour as the energy is changed.

It is clear that in principle this motion could lead to anomalous diffusion: in particular, as the energy is decreased the "periodic" motion within each cell should become more and more dominant in the dynamics of the particle. Obviously, the "random" diffusion from cell to cell is still in place, hence we only expect some weak departure from $H_H = 0.5$ as the mechanical energy is decreased. By inspection (see Fig. 2 in [18]) it is clear that the autocorrelation function of the velocity v has a visible negative tail (which is not at all surprising, remembering the importance of the periodic motion within each cell). We then expect that as the energy is decreased H_H should take on values smaller than 0.5. This is confirmed by the Hurst rescaled range analysis applied to the variable ξ, and shown in Fig. 7.

As clear from the discussion in the previous section, the fundamental question is whether the coefficient H_H should be related to H_D in situations of anomalous diffusion. We compare the quantity $R(\tau)$ (Hurst analysis) and $\langle v^2(\tau)\rangle^{1/2}$ (see Section 2) in Fig. 8: we used an energy value equal to -0.90, for which the dynamics is supposedly anomalous ($H_H = 0.43$). The clear parallelism between the two curves at large times establishes that the diffusion is indeed anomalous as suggested by the Hurst analysis.

4.5 Standard map

We now present the results obtained in a discrete model, i.e. for the standard map

$$
\begin{aligned}
x_{t+1} &= x_t + \frac{K}{2\pi}\sin\theta_t \\
\theta_{t+1} &= \theta_t + x_{t+1} \pmod{2\pi}
\end{aligned}
\tag{38}
$$

The standard map is very convenient to test our interpretation of the Hurst rescaled analysis analysis: it has been recently shown [19] that for appropriate parameter values

within the chaotic regime anomalous diffusion should arise.

According to [19], the anomalous diffusion is caused by chaotic orbits sticking to critical tori encircling accelerator mode islands. For this reason the correlation function $C(t)$ should have the power law dependency of (18). We studied the map for the same values considered in [19], i.e., for $K = 3.86$, 6.4717, 6.9115 and 10.053: the dynamical variable we considered for the analysis is the quantity $\xi_t = x_{t+1} - x_t$. A standard diffusion behaviour is expected for $K = 3.86$ and 10.053, and an anomalous diffusion behaviour for the other two K values. That this is qualitatively the case is clearly shown in Fig. 9, where we have plotted $R(\tau)$ normalized to $\tau^{1/2}$ versus τ for the different K's: as expected, the curve is an horizontal line (standard diffusion) for $K = 3.86$ and 10.053. We would now like to understand whether for the anomalous diffusion case we have some correspondence between the Hurst rescaled range analysis and the numerical work of [19]. In the case $K = 6.9115$, in [19] it is reported that theory and numerical simulations lead to

$$\alpha \sim \frac{2}{3} \tag{39}$$

Let us insert (39) into (19). Adopting the notation of [19], i.e., $\zeta \equiv 2H_D$, we obtain

$$\zeta \sim \frac{4}{3} = 1.3333333... \tag{40}$$

According to [19] this prediction fits very well the result of the numerical calculation on diffusion.

We remark once more that for $K = 3.86$ and 10.053, values for which the authors of [19] observe a standard diffusion we obtain, with a very high degree of accuracy $H_H = 0.5$; and where ($K = 6.9115$ and 6.4717) anomalous diffusion is predicted we obtain a value of H_H significantly different from $H_H = 0.5$. Moreover, for $K = 6.9115$ we obtain $\zeta = 1.2330$, to be compared to $\zeta = 1.3333$ from (40).

Table 1 summarizes the situation. Note that the fourth column, ζ_{sim} denotes the result of the numerical simulation [5] (the only value there reported corresponds to $K = 6.9115$) and that the last column, $\zeta(H_H)$ reports the values of ζ corresponding to the value of H_H evaluated numerically (third column).

5 Conclusions

It must be pointed out that the Mandelbrot analysis, leading to $H_H = H_D$, is essentially based on the Central Limit Theorem, assumed to be valid even when anomalous diffusion occurs. The adoption of the Fokker-Planck treatment of Section 3 leads to the following time dependence of the x-distribution:

$$\sigma(x;t) = \frac{1}{(2\pi K t^{2H})^{1/2}} \exp\left(-\frac{x^2}{4Kt^{2H}}\right) \tag{41}$$

It is then evident that the moment $\langle x^n(t)\rangle$ rescales in time as t^{2H}. Since the Hurst rescaled range analysis refers to a quantity with the same dimension of x it is evident that it leads to

$$H_H = H = H_D \tag{42}$$

The functional form (41) suggests that in general, after an initial transient, the probability distribution $\sigma(x;t)$ should perhaps be described by the equation

$$\sigma(x;t) = \frac{1}{t^\beta} F\left(\frac{x}{t^\beta}\right) \tag{43}$$

If the rescaling of (43) applied, we would have that indeed in general $H_H = H_D$.

Let us now briefly discuss some possible forms of F. There are three possible conditions:

(i) $\beta = 1/2$, F is a Gaussian function of its argument. This is the standard diffusion process.

(ii) $\beta \neq 1/2$, F is a Gaussian function of its argument. This is the fractional Brownian motion process.

(iii) $\beta \neq 1/2$, F is not a Gaussian function of its argument. Note that this occurs for a Lévy stable process [20].

It must be pointed out, however, that from a physical point of view it is hard to imagine a diffusion process with a deterministic origin agreeing with (41), and thus falling under case (ii), even in the case $H > 0.5$. The reason is that as established by the theoretical analysis of Section 2, this anomalous behavior comes from an anomalously slow correlation function, namely the correlation function of (8) with $\alpha < 1$. In this physical situation, there is no hope to realize (41) as an effect of the Central Limit Theorem: the original process must be already Gaussian! In other words, if there existed Gaussian statistical processes leading to the slow decay of $C(t)$, then the anomalous diffusion would be compatible with the time rescaling of (41). In our opinion, this is the physical nature of the fractional Brownian motion of Mandelbrot. It is the long-time asymptotic limit of a Gaussian process with an anomalously slow correlation function.

We think that this situation might occur in statistical mechanics when the source of the Brownian motion, the statistical process x, refers to a physical condition characterized by a large number of degrees of freedom. However, in the last few years, there have been attempts to build statistical mechanics on chaos, without the joint action of a very large number of degrees of freedom [7, 8, 9]. In this physical situation x is a non-Gaussian statistical process and the Gaussian nature of diffusion stems from the action of the

Central Limit Theorem. If the process is not Gaussian, but it is fully chaotic, then the correlation function is exponential or, more generally, characterized by a well-defined time scale.

In the special case when chaos and ordered motion coexist, however, the dynamical behavior of the system becomes extremely more complex, and a correlation function with an inverse power law might occur. This implies the breakdown of the time scale separation between diffusion and microscopic dynamics, and the consequent breakdown of the Central Limit Theorem itself. In this physical condition (41) cannot apply.

Are there processes rescaling according to (iii), without implying the Gaussian assumption? We think that if an anomalous diffusion exists, then it is quite probable that it belongs to the class (iii). We are convinced that some of the processes examined in Section 4 belong to class (iii). If the rescaling in (43) with $\beta \neq 1/2$ holds true, then we conclude immediately that $H_H = \beta$. However, this special condition raises the intriguing question of whether or not $H_H = H_D$, in this case. The dynamical realization of the diffusion process is expressed by (9). Let us assume the X is the maximum possible value of ξ. It is then evident that at the time N the ξ distribution must be contained between x_m and $-x_m$, with $x_m = NX$. Now, let us imagine that there are theoretical reasons to expect that the x distribution is characterized by long tails with an inverse power law $1/x^m$. It is then evident that the rescaling of (43) cannot apply to the whole space. This might generate a discrepancy between H_H and H_D. Let us assume, for simplicity, that $x_m = At$. In such a case we get a rescaling of the same kind as (43) only for $|x_m/t| < A$. Thus the moments of the distribution rescale with a power law different from that leading to the time rescaling of (43).

We wonder if a possible discrepancy between the two coefficients might be derived using

the data already available for the standard map [19]. According to [19], the distribution rescales as in (43), with $\beta = 3/5$ for $K = 6.9115$. However, this distribution is truncated at the value $|y| = |x/t^{\beta}| = 1$. If we assume that H_H is determined by the rescaling of (43), we obtain

$$2H_H = 2\beta = 6/5 = 1.2 \tag{44}$$

thereby suggesting that the discrepancy between the numerical value $2H_H = 1.23330$, obtained in this paper, and the numerical value $2H_D = 1.3333333$, determined numerically by the authors of [19], might be due not to the inaccuracy of the direct calculation of H_D in [19] (notice that the calculation of H_H is expected to more accurate than that of H_D), but it might rather depend on the breakdown of the condition $H_H = H_D$, due to the non-Gaussian character of the distribution F of (43).

We shall address these questions in further investigations. For the time being we must limit ourselves to saying that the Hurst rescaled range analysis seems to be an efficient numerical technique to explore how a dynamical system approaches its long-time asymptotic limit, or, equivalently, which is the short time dynamics of that asymptotic idealization referred to by Mandelbrot [4, 5, 6] as fractional Brownian motion.

Acknowledgments

We warmly thank professor Jaczek Kowalski on many illuminating discussions on the possible difference between H_H and H_D. This work was partially supported by the EC under contract n. SC1-CT91-0697 (TSTS). One of us (B.J.W.) also thanks the Naval Air Warfare Center for partial support of this research [contract No. N62269-92-C-0548] and the Office of Naval Research [contract No. 73191].

References

[1] H.E. Hurst, Trans. Am. Soc. Civ. Eng. **116**, 770 (1951).

[2] H.E. Hurst, R.P. Black and Y.M. Simaika, Long Term Storage: An Experimental Study, Constable, London, 1965.

[3] J. Feder, Fractals, Plenum Press, New York, 1988.

[4] B.B. Mandelbrot and J.R. Wallis, Water Resour. Res. **5**, 260 (1969).

[5] B.B. Mandelbrot and J.W. van Ness, Siam Rev. **10**, 422 (1968).

[6] B.B. Mandelbrot, The Fractal Geometry of Nature, W.H. Freeman, New York, 1983.

[7] Gy. Trefán, P. Grigolini, and B.J. West, Phys. Rev. A **45**, 1249 (1992).

[8] M. Bianucci, R. Mannella, X. Fan, P. Grigolini, and B.J. West, Phys. Rev. E **47**, 1510 (1993).

[9] M. Bianucci, L. Bonci, Gy. Trefán, B.J. West, and P. Grigolini, Phys. Lett. A **174**, 377 (1993).

[10] B.J. West, H.J. Mackey and D. Chen, in *Patterns, Information and and Chaos in Neural Systems*, ed. by B.J. West, World Scientific, Singapore (1993).

[11] B.J. Alder and T.E. Wainwright, Phys. Rev. Lett. **18**, 968 (1967), Phys. Rev. A **1**, 18 (1970).

[12] The literature on this topic is huge. For this reason, we limit ourselves to quoting only the recent theoretical paper of reference [13], which provides the first microscopic

derivation of an inverse power law. The reader can find there also an extended set of references on the subject of slow decay.

[13] R. Dekeyser, H.H. Lee, Phys. Rev. B **43**, 8123 (1991); ibid. **43**, 8131 (1991).

[14] J.A. Novotny and E. Jacobsson, Biophys. J. **64**, 98 (1993).

[15] P. Grigolini, in "Noise in Nonlinear Dynamical Systems", F. Moss and P.V.E. McClintock editors, Cambridge University Press (1989), Volume 1, p.161.

[16] R. Zwanzig, J. Chem. Phys. **33**, 1338 (1960).

[17] I.I. Fedchenia, R. Mannella, P.V.E. McClintock, N.D. Stein and N.G. Stocks, Phys. Rev. A **46**, 4 (1992).

[18] B. Bagchi, R. Zwanzig and M.C. Marchetti, Phys. Rev. A **31**, 892 (1985).

[19] R. Ishizaki, T. Horita, T. Kobayashi and H. Mori, Prog. Theor. Phys. **85**, 1013 (1991).

[20] E.W. Montroll and B.J. West, in *Fluctuation Phenomena*, Eds. E.W. Montroll and J.L. Lebowitz, Elsevier Publishing Comp., North Holland Personal Library, 2nd ed. (1987).

Table caption

Table summarising the comparison between H_H from numerical simulations (third column) as function of the parameter K (first column). The power of the correlation function tail, from the simulations, is shown under the heading α. For the definition of ζ see text. Note that ζ_{sim} is the value computed in [18], and that $\zeta(H_H)$ is computed from our H_H values.

Figure captions

- Figure 1: Typical slow decaying correlation functions with a positive tail (solid line) and a negative tail (dashed line).

- Figure 2: the Hurst rescaled range analysis applied to the Brownian motion of 32. The labels indicate the resulting Hurst function $R(\tau)$, for different γ. The straight lines are best fits to the appropriate power law.

- Figure 3: The Hurst rescaled range analysis is applied to the Brownian motion of 33. The parameters values are: full line, $\Omega^2 = 10$, $\gamma = 1$; dashed line $\Omega^2 = 1$, $\gamma = 10$; and dot-dashed line $\Omega^2 = 1$, $\gamma = 1$.

- Figure 4: Comparison between the Hurst rescaled analysis (dashed line) and diffusion of the variable x ($= \langle x^2(\tau)\rangle^{1/2}$, see Section 2, solid line) in the system described by 42. Parameter values chosen are $\Omega^2 = 1$, $\gamma = 10$. Note the parallelism between the different curves at large τ times.

- Figure 5: The Hurst rescaled range analysis applied to the system described by 34: for $D \simeq 0.1$ chosen, $T_K \approx 30$, and $T_R \sim 1$ (see text). The boxed numbers are the best fit H_H values evaluated in the region around the arrow head.

- Figure 6: The Hurst rescaled range analysis is applied to the Lorenz model (35). The curves are drawn for increasing noise intensities (solid line, $D = 0$, dotted line, barely visible in the bottom left corner,$D = 10^{-6}$, small dashes line, $D = 10^{-5}$, large dashes line, $D = 10^{-4}$). The quantity plotted is $R(\tau)$ normalized to $\tau^{1/2}$ versus τ, and in case of $H_H = 0.5$ we should have an horizontal line.

- Figure 7: The Hurst rescaled range analysis is applied to the two-dimensional model of 36 and 37. From top to bottom the energy (in round brackets H_H) has the value -0.60 (0.49), -0.80 (0.50), -0.85 (0.49), -0.90 (0.43) and -0.95 (0.39). We plotted the quantity $R(\tau)$ normalized to $\tau^{1/2}$ versus τ, and in case of $H_H = 0.5$ we should have an horizontal line.

- Figure 8: Comparison between the Hurst rescaled range analysis (solid line) and diffusion ($= \langle v^2(\tau)\rangle^{1/2}$, see Section 2, dashed line) for the model of 36 and 37, and for a mechanical energy equal to -0.90 ($H_H = 0.43$). The parallelism between the curve proves the relevance of the Hurst analysis in the calculation of the diffusion at large times.

- Figure 9: The Hurst rescaled range is applied analysis of the standard map, 38. The quantity $R(\tau)$ normalized to $\tau^{1/2}$ versus τ is plotted, and in case of $H_H = 0.5$ horizontal line results. The different curves refer to different values of the parameter K (see text). We have: solid line, $K = 3.86$; small dashes line, $K = 6.4714$; large dashes line, $K = 6.9115$; and dot dashed line, $K = 10.053$. Standard diffusion is expected for $K = 3.86$ and $K = 10.053$.

Table I

K	α	H_H	ζ_{sim}	$\zeta\,(H_H)$
10.053	0.9483 ± 0.0058	0.50317 ± 0.0004		1.0624
6.9115	0.80188 ± 0.00888	0.61665 ± 0.00081	1.3333	1.2333
6.4717	0.88778 ± 0.00854	0.57652 ± 0.00066		1.15304
3.86	0.86957 ± 0.00443	0.50134 ± 0.000307		1.0268